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THE INTERACTION OF THERMAL RADIATION WITH FREE CONVECTION HEAT TRANSFER

R. D. CESS
State University of New York at Stony Brook

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Abstract—The dimensionless boundary-layer equations describing free convection of a radiation absorbing–emitting fluid illustrates that the problem may be treated as a singular perturbation problem. Specifically, this treatment applies if the parameter characterizing the relative importance of conduction versus radiation heat transfer within the fluid is small. For several typical gases this is shown to be the case. The singular perturbation problem is formulated for laminar free convection of a gray gas along a vertical, black, isothermal plate. An illustrative solution of the resulting equations is presented for second-order interactions between radiation and free convection.

NOMENCLATURE

c_p, specific heat at constant pressure;
$E_n(t)$, exponential integral,
$$\int_0^1 \mu^{n-2} \exp(-t/\mu)\,d\mu;$$
f, dimensionless stream function;
$\bar{f}$, $f/\sqrt{(2PrN)}$;
F, equal to f for $N = 0$;
g, acceleration of gravity;
Gr, Grashof number;
k, thermal conductivity;
N, $k\kappa/4\sigma T_\infty^3$;
Nu, Nusselt number, $q_{cw}x/k(T_w - T_\infty)$;
Pr, Prandtl number, ν/α;
q, local heat-transfer rate per unit area;
t, dummy variable of integration;
T, absolute temperature;
ΔT, $T_w - T_\infty$;
u, velocity component in x-direction;
v, velocity component in y-direction;
x, coordinate along plate surface;
y, coordinate normal to plate surface.

Greek symbols
α, thermal diffusivity, $k/\rho c_p$;
β, coefficient of thermal expansion;
Γ, $[(\theta_w^4 - 1)/(\theta_w - 1)]^{\frac{1}{4}}$;
κ, volumetric absorption coefficient;
η, $\bar{\tau}/\xi^{\frac{1}{4}}$;
θ, dimensionless temperature, T/T_∞;
θ_w, T_w/T_∞;
Θ, equal to θ for $N = 0$;
ν, kinematic viscosity;
ξ, $4\sigma\kappa T_\infty^3 x^{\frac{1}{2}}/\rho c_p(g\beta\,\Delta T)^{\frac{1}{2}}$;
ρ, density;
σ, Stefan–Boltzmann constant;
τ, optical coordinate, κy;
$\bar{\tau}$, $\tau/\sqrt{(2PrN)}$;
ψ, stream function.

Subscripts
c, conduction;
R, radiation;
w, plate surface;
δ, outer edge of boundary layer;
∞, ambient.

INTRODUCTION

CONVECTION phenomena involving fluids which absorb and emit thermal radiation is an area which has recently attracted considerable attention with respect to forced convection. There have not, however, been any studies made concerning combined free convection and radiation heat transfer. Since radiation interaction is most pronounced when convection heat transfer is small, it would appear that radiation could play a significant role in free convection problems involving absorbing–emitting fluids.

The present investigation considers laminar free convection of an absorbing–emitting fluid along a vertical flat plate. The method of solution is patterned after the singular perturbation solution presented in references [1] and [2] for forced convection. A general formulation of the singular perturbation problem is presented, and an illustrative solution of the resulting equations, applicable for small values of the convection–radiation interaction parameter, is given. The purpose of the investigation is primarily to illustrate how such interaction effects occur. Thus, a number of simplifying assumptions, such as a black plate and a gray fluid, are employed.

BASIC EQUATIONS

The theoretical model and coordinate system are illustrated in Fig. 1 for a heated plate, while for a cooled plate the y-coordinate is reversed. Laminar free convection of a constant-property fluid is assumed, and the surface temperature of the plate is taken to be uniform. In addition, it is assumed that the plate surface is black and that the absorbing–emitting fluid is gray and nonscattering.

In terms of the stream function ψ, defined by

$$u = \frac{\partial\psi}{\partial y}, \qquad v = -\frac{\partial\psi}{\partial x}$$

the boundary-layer forms of the momentum and energy equations are, respectively

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2} = \nu\frac{\partial^3\psi}{\partial y^3} + g\beta(T - T_\infty) \quad (1)$$

$$\frac{\partial\psi}{\partial y}\frac{\partial T}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} - \frac{1}{\rho c_p}\frac{\partial q_R}{\partial y} \quad (2)$$

where q_R denotes the radiation flux within the fluid in the y-direction. From [1]*

$$-\frac{\partial q_R}{\partial\tau} = 2\sigma T_\infty^4 E_2(\tau) + 2\sigma\int_0^\infty T^4(x, t)\, E_1(|\tau - t|)\, dt - 4\sigma T^4(x, \tau) \quad (3)$$

where $\tau = \kappa y$ is the optical transverse coordinate.

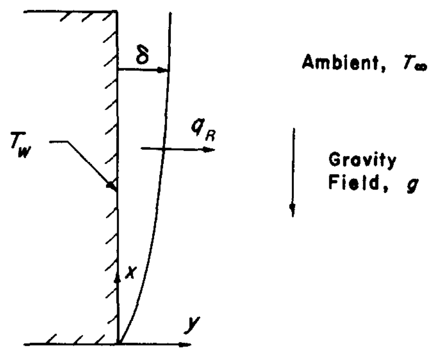

FIG. 1. Physical model and coordinate system.

It will be convenient to recast equations (1) and (2) in dimensionless form, and dimensionless quantities will be defined as

$$\xi = \frac{4\sigma\kappa T_\infty^3 x^{\frac{1}{2}}}{\rho c_p (g\beta\,\Delta T)^{\frac{1}{2}}}, \qquad N = \frac{k\kappa}{4\sigma T_\infty^3}$$

$$\psi(x, y) = \left(\frac{g\beta\,\Delta T x}{\kappa}\right)^{\frac{1}{2}} f(\epsilon, \tau),$$

$$T(x, y) = T_\infty\Theta(\xi, \tau).$$

For free convection the velocity component u is of the order

$$u = 0[\sqrt{(g\beta\,\Delta T x)}]$$

and consequently

$$\xi \sim \frac{4\sigma\kappa T_\infty^4 x}{\rho c_p u T_\infty}.$$

Thus ξ denotes the relative role of radiation to convection heat transfer. The parameter N is in turn a measure of the importance of conduction versus radiation within the fluid [1].

* For fluids with an index of refraction other than unity, σ is replaced by $n^2\sigma$ [1].

In terms of $f(\xi, \tau)$, the velocity components are

$$u = \left(\frac{\rho c_p g \beta \, \Delta T}{4 \sigma \kappa T_\infty^3}\right) \xi \frac{\partial f}{\partial \tau} \quad (4)$$

$$v = -\left(\frac{2 \sigma T_\infty^3}{\rho c_p}\right)\left(\frac{1}{\xi} f + \frac{\partial f}{\partial \xi}\right) \quad (5)$$

while equations (1) and (2) transform to

$$\left(\frac{\partial f}{\partial \tau}\right)^2 + \xi \frac{\partial f}{\partial \tau} \frac{\partial^2 f}{\partial \xi \partial \tau} - f \frac{\partial^2 f}{\partial \tau^2} - \xi \frac{\partial f}{\partial \xi} \frac{\partial^2 f}{\partial \tau^2} = 2 Pr\, N \xi \frac{\partial^3 f}{\partial \tau^3} + 2 \frac{\theta - 1}{\theta_w - 1} \quad (6)$$

$$\frac{\partial f}{\partial \tau} \frac{\partial \theta}{\partial \xi} - \frac{1}{\xi} f + \frac{\partial f}{\partial \xi} \frac{\partial \theta}{\partial \tau} = 2N \frac{\partial^2 \theta}{\partial \tau^2} + \theta_w^4 E_2(\tau) + \int_0^\infty \theta^4(\xi, t) E_1(|\tau - t|) \, dt - 2\theta^4(\xi, \tau). \quad (7)$$

Inspection of equations (6) and (7) shows that the highest derivative in each equation is multiplied by the parameter N, and if this parameter is small, the problem reduces to a singular perturbation problem.

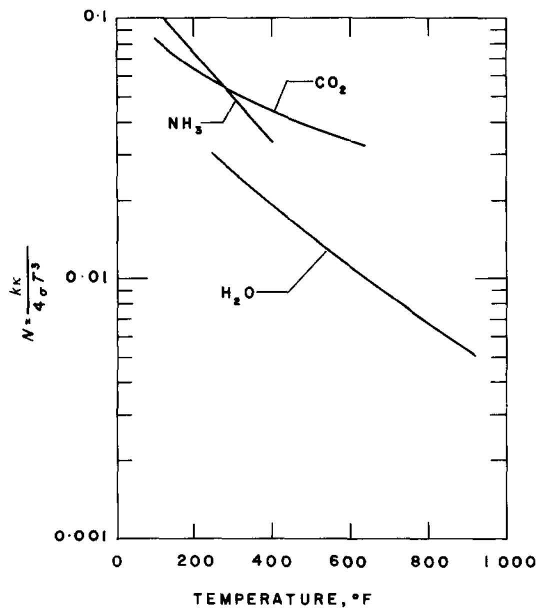

FIG. 2. Values of N for carbon dioxide, water vapor and ammonia at one atmosphere.

To gain some insight into typical magnitudes of N, values of N are illustrated in Fig. 2 for water vapor, carbon dioxide and ammonia, where N has been calculated using the Planck mean absorption coefficient. It is seen that, at least for these gases, the magnitude of N is much less than unity. A meaningful and physically practical approach to the present problem would thus be to consider the limiting case $N \ll 1$.

FORMULATION FOR $N \ll 1$

In formulating the present problem as a singular perturbation problem, an approximation for $N \ll 1$ is obtained by setting $N = 0$ in equations (4) and (5). Letting F and Θ denote f and θ for the condition $N = 0$, then

$$\left(\frac{\partial F}{\partial \tau}\right)^2 + \xi \frac{\partial F}{\partial \tau} \frac{\partial^2 F}{\partial \xi \partial \tau} - F \frac{\partial^2 F}{\partial \tau^2} - \xi \frac{\partial F}{\partial \xi} \frac{\partial^2 F}{\partial \tau^2} = 2\left(\frac{\Theta - 1}{\theta_w - 1}\right) \quad (8)$$

$$\frac{\partial F}{\partial \tau} \frac{\partial \Theta}{\partial \xi} - \left(\frac{1}{\xi} F + \frac{\partial F}{\partial \xi}\right) \frac{\partial \Theta}{\partial \tau} = \theta_w^4 E_2(\tau) + \int_0^\infty \Theta^4(\xi, t) E_1(|\tau - t|) \, dt - 2\Theta^4(\xi, \tau). \quad (9)$$

With the exception of optically thick radiation ($\tau \gg 1$), equation (9) cannot satisfy continuity of temperature at the plate surface, and an inner solution, which includes the highest derivatives, is additionally required within a thin region near the plate surface. This inner solution is also necessary in order to satisfy the no-slip velocity condition, although the no-slip condition may be shown to be redundant for $\tau \gg 1$.* Thus, since equations (8) and (9) adequately describe the problem for $\tau \gg 1$, one is concerned with an inner solution for $\tau \leqslant 0(1)$, where τ refers to the penetration depth of the *entire* temperature field. Furthermore, in a manner

* For $\tau \gg 1$ the problem is mathematically analogous to that of pure free convection for the limit $Pr \to 0$. In this case the no-slip condition is not imposed when solving for the temperature field [3].

analogous to forced convection [1], this in turn implies $\xi \leqslant 0(1)$. With little loss in generality, the restriction $Pr = 0(1)$ will also be imposed.

The present problem has thus been reduced to one involving inner and outer solutions, with equations (8) and (9) describing the outer solution. To apply equations (6) and (7) to the inner region, consider the change of variables

$$\bar{\tau} = \frac{\tau}{\sqrt{(2Pr\,N)}}, \qquad \bar{f} = \frac{f}{\sqrt{(2Pr\,N)}}$$

and equations (6) and (7) become

$$\left(\frac{\partial \bar{f}}{\partial \bar{\tau}}\right)^2 + \xi \frac{\partial \bar{f}}{\partial \bar{\tau}} \frac{\partial^2 \bar{f}}{\partial \xi \partial \bar{\tau}} - \bar{f} \frac{\partial^2 \bar{f}}{\partial \bar{\tau}^2} - \xi \frac{\partial \bar{f}}{\partial \xi} \frac{\partial^2 \bar{f}}{\partial \bar{\tau}^2} = \xi \frac{\partial^3 \bar{f}}{\partial \bar{\tau}^2} + 2\,\frac{\theta - 1}{\theta_w - 1} \tag{10}$$

$$\frac{\partial \bar{f}}{\partial \bar{\tau}} \frac{\partial \theta}{\partial \xi} - \left(\frac{1}{\xi} \bar{f} + \frac{\partial \bar{f}}{\partial \xi}\right) \frac{\partial \theta}{\partial \bar{\tau}} = \frac{1}{Pr} \frac{\partial^2 \theta}{\partial \bar{\tau}^2} \tag{11}$$
$$+ \theta_w^4\, E_2[\bar{\tau}\sqrt{(2Pr\,N)}] + \int_0^\infty \theta^4(\xi, t)\, E_1[|\bar{\tau}\sqrt{(2Pr\,N)} - t|]\, dt - 2\theta^4(\xi, \tau).$$

With this change of variables, the highest derivative in both equation (10) and equation (11) is now of the same magnitude as the other terms in each equation. Furthermore, it is evident that these highest derivatives are of importance within a region of thickness $\bar{\tau} = 0(1)$; that is, the optical thickness of the inner region (or boundary layer) is $\tau_\delta = 0[\sqrt{(2Pr\,N)}]$. Clearly the inner region is optically thin. Equation (11) may be further simplified by noting that with $\bar{\tau} = 0(1)$

$$E_2[\bar{\tau}\sqrt{(2Pr\,N)}] = 1 + 0[\sqrt{(2Pr\,N)}] \tag{12}$$

and

$$E_1[|\bar{\tau}\sqrt{(2Pr\,N)} - t|] = E_1(t) + 0[\sqrt{(2Pr\,N)}].$$

From this last expression, and noting that $\theta = \Theta$ for $\tau > \tau_\delta$, one has

$$\int_0^\infty \theta^4(\xi, t)\, E_1[|\bar{\tau}\sqrt{(2Pr\,N)} - t|]\, dt = \int_0^\infty \theta^4(\xi, t)\, E_1(t)\, dt = \int_0^\infty \Theta(\xi, t)\, E_1(t)\, dt$$
$$+ \int_0^{\tau_\delta} [\theta(\xi, t) - \Theta(\xi, t)]\, E_1(t)\, dt$$
$$= \int_0^\infty \Theta(\xi, t)\, E_1(t)\, dt + 0[\sqrt{(2Pr\,N)}]. \tag{13}$$

Employing equations (12) and (13), equation (11) reduces to

$$\frac{\partial \bar{f}}{\partial \bar{\tau}} \frac{\partial \theta}{\partial \xi} - \left(\frac{1}{\xi} \bar{f} + \frac{\partial \bar{f}}{\partial \xi}\right) \frac{\partial \theta}{\partial \bar{\tau}} = \frac{1}{Pr} \frac{\partial^2 \theta}{\partial \bar{\tau}^2} + \theta_w^4 + \int_0^\infty \Theta^4(\xi, t)\, E_1(t)\, dt - 2\theta^4(\xi, \tau). \tag{14}$$

The outer region is thus described by equations (8) and (9) and the inner region by equations (10) and (14). It remains to specify boundary and matching conditions.

For the outer region the boundary conditions are

$$T \to T_\infty, \qquad u \to 0; \qquad \tau \to \infty$$

or, with reference to equation (4)

$$\Theta(\xi, \infty) = 1, \qquad \frac{\partial F(\xi, \infty)}{\partial \tau} = 0. \tag{15}$$

For the inner region it is required that

$$T = T_w, \qquad u = v = 0; \qquad y = 0$$

and noting equations (4) and (5)

$$\theta(\xi, 0) = \theta_w, \qquad \bar{f}(\xi, 0) = \frac{\partial \bar{f}(\xi, 0)}{\partial \bar{\tau}} = 0. \tag{16}$$

General principles of asymptotic matching are discussed by Van Dyke [4]. For the present case it is sufficient to state that the inner solution for large $\bar{\tau}$ must match the outer solution written in terms of the inner variable. Considering temperature, for example, the outer solution is written as $\Theta[\xi, \bar{\tau}\sqrt{(2Pr\,N)}]$, and asymptotic matching requires that

$$\theta(\xi, \infty) = \Theta(\xi, 0). \tag{17}$$

With respect to matching of the velocity components u and v, a similar procedure yields

$$\frac{\partial \bar{f}(\xi, \infty)}{\partial \bar{\tau}} = \frac{\partial F(\xi, 0)}{\partial \tau} \tag{18}$$

$$F(\xi, 0) = 0. \tag{19}$$

It is evident that the outer solution is completely described by equations (8) and (9) together with conditions (15) and (19), and that these are independent of the inner solution. Thus, the procedure is to first obtain the outer solution for $F(\xi, \tau)$ and $\Theta(\xi, \tau)$, and then, making use of these results, to evaluate the inner solution from equations (10) and (14) together with conditions (16), (17) and (18). In this regard, the integral appearing in equation (14) may be rephrased as

$$\int_0^\infty \Theta^4(\xi, t)\, E_1(t)\, \mathrm{d}t = 2\Theta^4(\xi, 0) - \theta_w^4 + \left(\frac{\partial F}{\partial \tau}\frac{\partial \Theta}{\partial \xi}\right)_{\tau=0} \tag{20}$$

by setting $\tau = 0$ in equation (9) and noting equation (19).

To express the surface heat transfer in terms of the temperature functions $\theta(\xi, \bar{\tau})$ and $\Theta(\xi, \tau)$, it will be convenient to consider separately the conduction and radiation contributions. If use is made of the conventional definition of the Nusselt number, then

$$Nu = \frac{q_{cw}x}{k(T_w - T_\infty)} = -\frac{x}{(T_w - T_\infty)}\left(\frac{\partial T}{\partial y}\right)_{y=0}$$

and in dimensionless form the conduction heat transfer is given as

$$\frac{Nu}{(Gr)^{\frac{1}{4}}} = -\frac{\xi^{\frac{1}{4}}}{\sqrt{2}(\theta_w - 1)}\left(\frac{\partial \theta}{\partial \bar{\tau}}\right)_{\bar{\tau}=0} \tag{21}$$

Following [1, 2], the surface radiation heat transfer is expressed by

$$\frac{q_{Rw}}{\sigma T_\infty^4} = \theta_w^4 - 2\int_0^\infty \Theta^4(\xi, t)\, E_2(t)\, \mathrm{d}t \tag{22}$$

with terms of $0\,[\sqrt{(2Pr\, N)}]$ deleted.

In summary, the present formulation for $N \ll 1$ has reduced the initial integro-differential equations (6) and (7) to two simpler systems of equations; equations (8) and (9), which are still integro-differential but are of lower order; and equations (10) and (14), which are solely differential equations. The solution of these reduced equations is, however, still not an easy task. In the following an illustrative solution is presented which is restricted to small values of the interaction parameter ξ.

SOLUTION FOR SMALL ξ

As an example of the preceding formulation for $N \ll 1$, a solution, which is restricted to second-order interactions of radiation with convection, will be illustrated.

Analysis

Letting

$$\Gamma = \left(\frac{\theta_w^4 - 1}{\theta_w - 1}\right)^{\frac{1}{3}}$$

solutions of equations (8) and (9) may be expressed as

$$F(\xi, \tau) = \Gamma\xi^{\frac{1}{3}}[F_0(\tau) + F_1(\tau)\,\xi^{\frac{2}{3}} + \ldots] \tag{23}$$

$$\Theta(\xi, \tau) = 1 + (\theta_w - 1)\,\Gamma^2\xi^{\frac{2}{3}}[G_0(\tau) + G_1(\tau)\,\xi^{\frac{2}{3}} + \ldots] \tag{24}$$

for which the functions $F_0(\tau)$, $F_1(\tau)$, $G_0(\tau)$, and $G_1(\tau)$ are described by the ordinary differential equations

$$\left.\begin{aligned} F_0F_0'' - (F_0')^2 - \tfrac{3}{2}G_0 \\ 2F_0G_0' - F_0'G_0 = -\tfrac{3}{2}E_2(\tau) \end{aligned}\right\} \tag{25}$$

$$\left.\begin{aligned} F_0F_1'' - \tfrac{5}{2}F_0'F_1' + \tfrac{3}{2}F_0''F_1 &= -\tfrac{3}{2}G_1 \\ F_0G_1' - F_0'G_1 &= \tfrac{1}{2}F_1'G_0 - \tfrac{3}{2}F_1G_0' \\ &+ \frac{3}{\Gamma}\left[2G_0 - \int_0^\infty G_0(t)\, E_1(|\tau - t|)\, \mathrm{d}t\right]. \end{aligned}\right\} \tag{26}$$

The boundary conditions follows from equations (15) and (19) to be

$$\left.\begin{aligned} F_0(0) = F_1(0) = 0 \\ G_0(\infty) = G_1(\infty) = F_0'(\infty) = F_1'(\infty) = 0. \end{aligned}\right\} \tag{27}$$

Turning next to the inner region, solutions of equations (10) and (14) may be written as

$$\bar{f}(\xi, \bar{\tau}) = 2\xi^{\frac{1}{2}}[f_0(\eta) + f_1(\eta)\,\Gamma\xi^{\frac{1}{3}} + f_2(\eta)\,\Gamma^2\xi^{\frac{2}{3}} + \ldots] \quad (28)$$

$$\theta(\xi, \bar{\tau}) = 1 + (\theta_w - 1)\,[\theta_0(\eta) + \theta_1(\eta)\,\Gamma\xi^{\frac{1}{3}} + \theta_2(\eta)\,\Gamma^2\xi^{\frac{2}{3}} + \ldots] \quad (29)$$

where

$$\eta = \frac{\bar{\tau}}{\xi^{\frac{1}{2}}} = \left(\frac{g\beta\,\Delta T}{4v^2}\right)^{\frac{1}{4}} \frac{y}{x^{\frac{1}{4}}}.$$

One may note that η is the similarity variable for no radiation interaction; that is, equations (28) and (29) constitute an expansion about the no-interaction solution, and this is compatible with the physical interpretation of ξ. Upon substituting equations (28) and (29) into equations (10) and (14), one obtains the ordinary differential equations

$$\left.\begin{aligned} & f_0''' + 3f_0f_0'' - 2(f_0')^2 = -\theta_0 \\ & \times\ \frac{1}{Pr}\theta_0'' + 3f_0\theta_0' = 0 \end{aligned}\right\} \quad (30)$$

$$\left.\begin{aligned} & f_1''' + 3f_0f_1'' - \tfrac{14}{3}f_0'f_1' + \tfrac{11}{3}f_0''f_1 = -\theta_1 \\ & \frac{1}{Pr}\theta_1'' + 3f_0\theta_1' - \tfrac{2}{3}f_0'\theta_1 = -\tfrac{11}{3}f_1\theta_0' \end{aligned}\right\} \quad (31)$$

$$\left.\begin{aligned} & f_2''' + 3f_0f_2'' - \tfrac{16}{3}f_0'f_2' + \tfrac{13}{3}f_0''f_2 \\ & \qquad = -\theta_2 + \tfrac{8}{3}(f_1')^2 - \tfrac{11}{3}f_1f_1'' \\ & \frac{1}{Pr}\theta_2'' + 3f_0\theta_2' - \tfrac{4}{3}f_0'\theta_2 = -\tfrac{11}{3}f_1\theta_1' \\ & \qquad - \tfrac{13}{3}f_2\theta_0' + \tfrac{2}{3}f_1'\theta_1. \end{aligned}\right\} \quad (32)$$

From equations (16), (17), and (18) together with (23) and (24), the boundary conditions are

$$\left.\begin{aligned} & f_0(0) = f_1(0) = f_2(0) = f_0'(0) = f_1'(0) = f_2'(0) = 0 \\ & \theta_0(0) = 1, \quad \theta_1(0) = \theta_2(0) = 0 \\ & f_0'(\infty) = 0, \quad f_1'(\infty) = \tfrac{1}{2}F_0'(0), \quad f_2'(\infty) = 0 \\ & \theta_0(\infty) = \theta_1(\infty) = 0, \quad \theta_2(\infty) = G_0(0). \end{aligned}\right\}$$

Note that with restriction to the terms considered in equations (28) and (29), knowledge of $F_1(\tau)$ and $G_1(\tau)$ is not required.

As would be expected, the functions $f_0(\eta)$ and $\theta_0(\eta)$ correspond to the free convection boundary layer in the absence of radiation interaction. Before proceeding, it is of interest to discuss the separate mechanisms by which radiation alters the free convection boundary layer (i.e. the inner region), and to illustrate how each of these enter into the expansions in powers of $\xi^{\frac{1}{3}}$ of equations (28) and (29). There are three such mechanisms, and these are as follows:

(1) The induced velocity at the outer edge of the boundary layer. From equations (4), (18) and (23), it is seen that this is a first-order radiation effect, since it initially appears as a term of order $\xi^{\frac{1}{3}}$.

(2) The variable temperature $\Theta(\xi, 0)$ imposed at the outer edge of the boundary layer. Since this first occurs as a $\xi^{\frac{2}{3}}$ term, as may be noted from equation (24), it is consequently a second-order radiation effect.

(3) The absorption and emission of radiant energy within the boundary layer. From equations (14) and (24), it is found that absorption and emission correspond to terms of order ξ, and this third-order effect does not appear in the present second-order analysis.

It should be noted that the temperature ratio $\theta_w = T_w/T_\infty$ does not appear in equations (31) and (32). If, however, third-order terms were retained in equations (28) and (29), the nonlinear absorption-emission term would introduce θ_w into the differential equations describing $f_3(\eta)$ and $\theta_3(\eta)$.

In terms of the present second-order analysis, the conduction heat transfer is found by combining equations (21) and (29), so that

$$\frac{Nu}{(Gr)^{\frac{1}{4}}} = -\frac{1}{\sqrt{2}}[\theta_0'(0) + \theta_1'(0)\,\Gamma\xi^{\frac{1}{3}} + \theta_2'(0)\,\Gamma^2\xi^{\frac{2}{3}} + \ldots]. \quad (33)$$

With regard to radiation transfer from the plate surface, one finds that

$$\frac{q_{Rw}}{\sigma(T_w^4 - T_\infty^4)} = 1 - \frac{8}{\Gamma}\xi^{\frac{2}{3}} \int_0^\infty G_0(t)E_2(t)\,dt + \ldots \quad (34)$$

as given by equations (22) and (24).

Numerical results

The present second-order solution requires the integration of equations (25), (31), and (32), and this has been accomplished numerically on an IBM 1620 computer. In the case of equations (25) a backward integration was employed, since two of the three boundary conditions are located at infinity. The backward integration was additionally necessary since equations (25) indicate a possible singularity at the origin. Figure 3 illustrates F_0' and G_0, and the quantities $F_0'(0)$

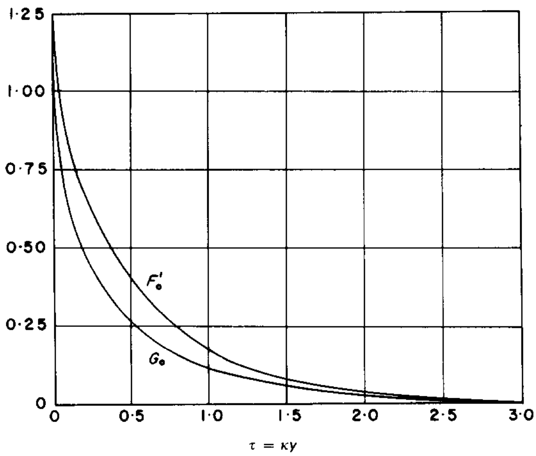

FIG. 3. The functions F_0 and G_0.

and $G_0(0)$, which appear in the boundary conditions for equations (31) and (32), were found to have the values

$$F_0'(0) = 1{\cdot}23, \qquad G_0(0) = 1{\cdot}01.$$

It should be noted that equations (25) are independent of Prandtl number.

Equations (31) and (32) have been solved numerically for $Pr = 1{\cdot}0$ using a conventional forward integration together with the tables of $f_0(\eta)$ and $\theta_0(\eta)$ given in [5]. These solutions are illustrated in Figs. 4 and 5, and the pertinent numerical results are

$$\theta_0'(0) = -0{\cdot}567, \qquad \theta_1'(0) = -0{\cdot}072, \qquad \theta_2'(0) = 0{\cdot}091. \quad (35)$$

Now, to evaluate the surface radiation heat transfer, the result

$$\int_0^\infty G_0(t)\,E_2(t)\,dt = 0{\cdot}167$$

was obtained by numerical integration, and from equation (34)

$$\frac{q_{Rw}}{\sigma(T_w^4 - T_\infty^4)} = 1 - \frac{1{\cdot}34}{\Gamma}\xi^{\frac{2}{3}} + \ldots \quad (36)$$

The first term in this equation simply represents radiation exchange between the plate surface and an infinite isothermal gas at temperature T_∞, since the emissivity of an infinite isothermal gas is unity. The second term in equation (36) represents a first-order correction due to the fact that the gas is actually nonisothermal.

On combining equations (33) and (35), the conduction heat transfer is described by

$$\frac{Nu}{(Gr)^{\frac{1}{4}}} = 0{\cdot}401 + 0{\cdot}051\Gamma\xi^{\frac{1}{3}} - 0{\cdot}064\Gamma^2\xi^{\frac{2}{3}} + \ldots \quad (37)$$

for $Pr = 1{\cdot}0$. The first term in this expression denotes free convection in the absence of any radiation interaction. The first-order interaction term, $0{\cdot}051\Gamma\xi^{\frac{1}{3}}$, is the result of the induced motion at the outer edge of the free convection boundary layer, and this "forced convection" effect results in an increase in convection heat transfer. The second-order interaction term in equation (37) includes both the second-order induced motion effect together with the first-order influence of the variable temperature imposed at the outer edge of the boundary layer.

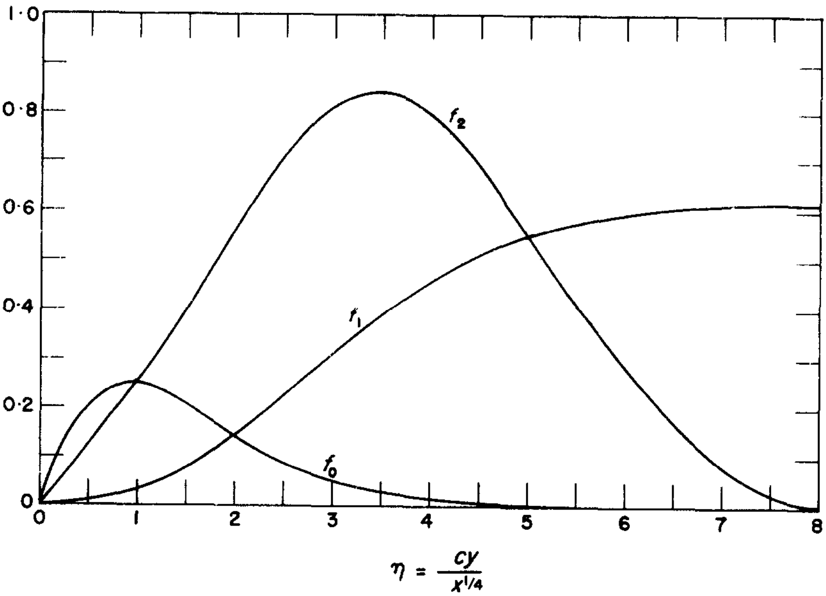

FIG. 4. The functions f'_0, f'_1, and f'_2 for $Pr = 1$.

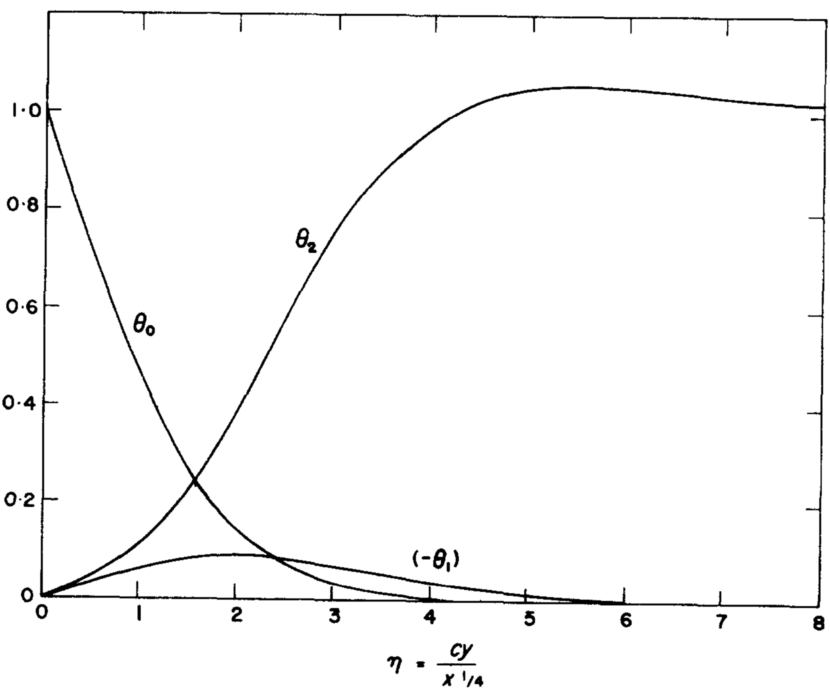

FIG. 5. The functions θ_0, θ_1, and θ_2 for $Pr = 1$.

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Résumé—Les équations sans dimensions de la couche limite qui décrivent la convection naturelle d'un fluide absorbant er émetteur montre que le problème peut être traité comme un problème de perturbation singulière.

Ce traitement s'applique convenablemant si le paramètre qui caractérise l'importance relative de la conduction par rapport au transport de chaleur par rayonnement dans le fluide, est petit. On a montré que c'est le cas pour plusieurs gaz typiques.

Le problème de perturbation singulière est formulé pour la convection naturelle laminaire d'un gaz gaz gris le long d'une plaque verticale, noire et isotherme. Une solution des équations résultantes est présentée comme exemple pour des interactions du second ordre entre le rayonnement et la convection naturelle.

Zusammenfassung—Die dimensionslosen Grenzschichtgleichungen, welche die freie Konvektion eines strahlungsabsorbierenden emittierenden Mediums beschreiben, zeigen, dass das Problem als ein singuläres störungsproblem behandelt werden kann. Speziell trifft dies zu, wenn der Parameter, der die relative Bedeutung des Wärmestransports durch Leitung gegenüng gegenüber der Strahlung charakterisiert, klein ist. Für verschiedene typische Gase wird das bestätigt. Das singuläre Strörungsproblem wird formuliert für die laminare freie Konvektion eines grauen Gases an einer senkrechten, schwarzen, isothermen Platte. Eine anschauliche Lösung der resultierenden Gleichungen ist angegeben für Wechselwirkungen zweiter Ordnung zwischen Strahlung und freier Konvektion.

Аннотация—Анализ безразмерных уравнений пограничного слоя при свободной конвекции в излучающе-поглощающей среде показывает, что задачи такого типа можно решать методом единичного возмущения. В частности, этот метод применим для случая низких значений параметра, характеризующего относительное соотноюение теплопроводности и лучистого теплообмена. Справедливост этого показана для нескольких типичных газов. Сформулирована задача о единичном возмущении для ламинарной свободной конвекции в серой газообразной среде вдоль вертикальной черной изотермической пластины. В качестве примера привододится решение полученных уравнений для взаимодействий излучения и свободной конвекции второго порядка.